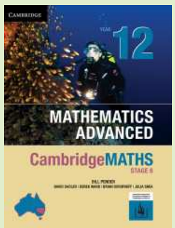
NORMANHURST BOYS HIGH SCHOOL

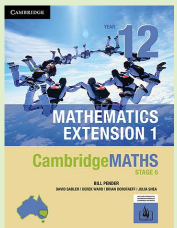
MATHEMATICS ADVANCED (INCORPORATING EXTENSION 1/2) YEAR 12 COURSE

Topic summary and exercises:

Ⓐ Ⓧ1 Ⓧ2 **Modelling Financial Situations**

With references to





Name: ....

Initial version by H. Lam, September 2014 (Applications of geometric series).
Major revisions July 2020 for Mathematics Advanced. Last updated November 22, 2021.
Various corrections by students & members of the Department of Mathematics at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under 🅭 CC BY 2.0.

Symbols used

⚠ Beware! Heed warning.

Enrichment - not necessarily in the syllabus, or any formal assessment.

(S2) Mathematics Standard 2 common content

(A) Mathematics Advanced content.

(X1) Mathematics Extension 1 content.

Literacy: note new word/phrase.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

MA12-2 models and solves problems and makes informed decisions about financial situations using mathematical reasoning and techniques

MA12-4 applies the concepts and techniques of arithmetic and geometric sequences and series in the solution of problems

Syllabus subtopics

MA-M1 Modelling Financial Situations

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 12 Advanced* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019a) or *CambridgeMATHS Year 12 Extension* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019b) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Section 1

Geometric series and logarithms

Laws/Results

Sum of a geometric series to n terms:

$$S_n = \dots\dots\dots\dots\dots\dots$$

Important note

When solving inequalities involving exponentials and using logarithms, be careful of $\log a$ where $|a| < 1$.

Further exercises

Only complete as many as required.

(A) **Ex 8A** Pender et al. (2019a)
- Q6-18

(A) **Ex 8B** Pender et al. (2019a)
- Q7-10

(X1) **Ex 14A** Pender et al. (2019b)
- Q6-14

(X1) **Ex 14B** Pender et al. (2019b)
- Q6-11

Section 2

S2 Compound Interest

Learning Goal(s)

Knowledge
How to solve

Skills
Careful algebraic manipulation

Understanding
Compound interest, effective interest rate

By the end of this section am I able to:

34.2 Solve compound interest problems involving financial decisions, including a home loan, a savings account, a car loan or superannuation

34.3 Use geometric sequences to model and analyse practical problems involving exponential growth and decay

- Calculate the effective annual rate of interest and use results to compare investment returns and cost of loans when interest is paid or charged daily, monthly, quarterly or six-monthly.

Definition 1

Future value (FV) of an investment is the total value of the investment at the end of the term of investment, including all contributions and the interest earned.

Definition 2

Present value (PV) of an investment is the is the single amount, which if invested at the same rate for the same period, would give that future value.

(Can also be understood as the .... value)

Laws/Results

Generally,

$$FV = PV + I$$

where I is the total amount of interest earned.
Compound interest formula

$$FV = PV(1+r)^n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

FV - Future value of the loan or amount (final balance)

PV - Present value of the loan or principal (initial quantity of money)

r - Rate of interest per compounding time period expressed as a decimal (or a fraction)

n - Number of compounding time periods

A - Amount

Important note

⚠ Remember to divide the interest rate *per annum* by the appropriate factor to place it on the time unit as the compounding period.

- Write the interest rate and associated unit *as a decimal* e.g. 0.025 p.a., 0.025 p.m.

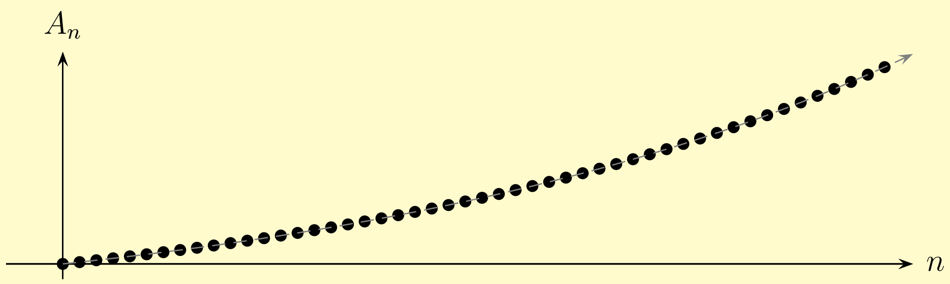
Definition 3

Future value interest factor (FVIF) the amount to multiply the present value by, to obtain the future value.

For a compound interest calculation: $FV = PV\overbrace{(1+r)^n}^{\text{FVIF}}$

Laws/Results

- General graph: increasing ....



  - More frequent .... will result in a faster .... of interest.

2.1 Simple calculations

Example 1

[2003 General Mathematics HSC Q16] Pauline calculates the present value (N) of an annuity. The interest rate is 4% per annum, compounded monthly. In five years the future value will be \$100 000.

Which calculation will result in the correct answer?

(A) $N = \dfrac{100\,000}{(1+0.04)^5}$

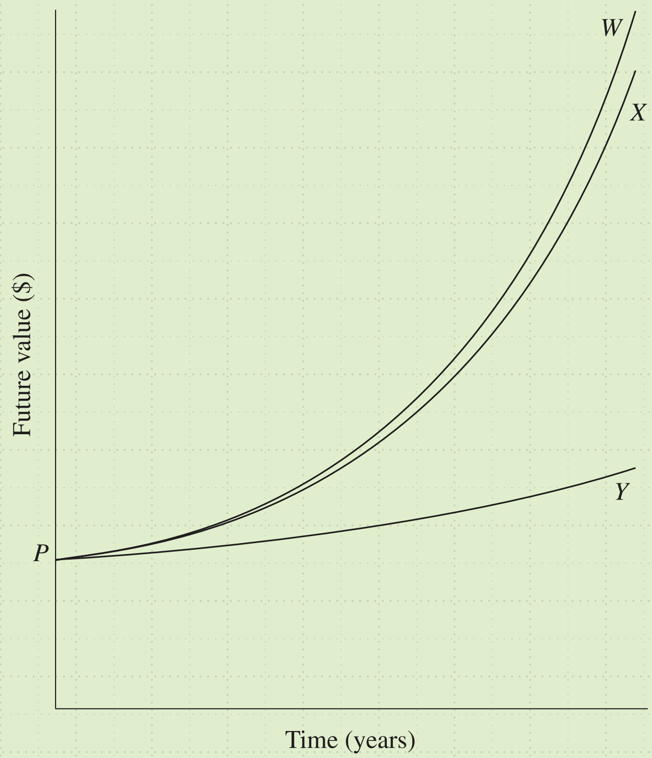
(B) $N = \dfrac{100\,000}{(1+0.04\div 12)^5}$

(C) $N = \dfrac{100\,000}{(1+0.04)^{60}}$

(D) $N = \dfrac{100\,000}{(1+0.04\div 12)^{60}}$

Example 2

[2020 Adv HSC Sample Q8/2019 Mathematics Standard 2 HSC Q13] The graphs show the future values over time of $\$P$, invested at three different rates of compound interest.



Which of the following correctly identifies each graph?

(A)

W	5% pa, compounding annually
X	10% pa, compounding annually
Y	10% pa, compounding quarterly

(B)

W	5% pa, compounding annually
X	10% pa, compounding quarterly
Y	10% pa, compounding annually

(C)

W	10% pa, compounding quarterly
X	10% pa, compounding annually
Y	5% pa, compounding annually

(D)

W	10% pa, compounding annually
X	10% pa, compounding quarterly
Y	5% pa, compounding annually

2.2 Table of interest factors for compound interest

Definition 4

Future value interest factor for a \$1 investment (compound interest) is

$$(1+r)^n$$

(which may need adjustment for compounding periods other than annually)

Example 3

[**2011 General Mathematics HSC Q23**] (2 marks) An amount of \$5 000 is invested at 10% per annum, compounded six-monthly.

Compounded values of \$1

Period	**Interest rate per period**				
	1%	*5%*	*10%*	*15%*	*20%*
1	1.010	1.050	1.100	1.150	1.200
2	1.020	1.103	1.210	1.323	1.440
3	1.030	1.158	1.331	1.521	1.728
4	1.041	1.216	1.464	1.750	2.074
5	1.051	1.276	1.611	2.011	2.488
6	1.062	1.340	1.772	2.313	2.986

Use the table to find the value of this investment at the end of three years.

2.3 Effective annual interest rate

Fill in the spaces

- The .... of compounding needs to be taken into effect when dealing with financial contracts.
- The .... affects the interest accrued.

Definition 5

Nominal interest rate is an annual rate that *does not* take the compounding frequency into account.

- Compounding periods per annum must to be specified, and the nominal rate divided by the number of compounding periods.

Definition 6

Effective (annualised) interest rate is converted from the nominal interest rate into a per annum interest rate. Mostly calculated based on the principal of \$1, and re-annualised back to 1 year.

Important note

What about the 'comparison rate'? See Mozo

- What assumptions are made?
- It gets tricky!

Example 4

(a) $1 000 is invested at 6% p.a., compounding half yearly.

- Calculating the future value of the investment after 1 year:

$$FV = \$1\,000 \times \left(1 + \ldots\ldots\ldots\right)^2 = \$1\,000 \times \left(\ldots\ldots\ldots\ldots\right)^2$$

$$= \$1\,000 \times \left(\ldots\ldots\ldots\ldots\ldots\ldots\right)$$

6% p.a. compounding half yearly, is *equivalent* to .... compounded annually.

- **Nominal rate:** ....
- **Effective rate:** ....

(b) $1 000 is now invested at 5.94% p.a. compounded monthly.

- Calculating the future value of the investment after 1 year:

$$FV = \$1\,000 \times \left(1 + \ldots\ldots\ldots\ldots\ldots\right)^{12} = \$1\,000 \times \left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right)^{12}$$

$$\approx \$1\,000 \times \left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right)$$

5.94% p.a. compounding monthly, is *equivalent* to .... compounded annually.

- **Nominal rate:** ....
- **Effective rate:** ....

(c) Which has a higher effective rate? 6% compounding half yearly, or 5.94% compounding monthly?

Laws/Results

The **effective interest rate** E, given a nominal rate r_{nom} compounding m times *per year*:

$$E = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Example 5

What is the effective interest rate for an investment if the nominal rate quoted is 15.75% p.a., compounding daily? **Answer:** 17.05

Example 6

[2006 Mathematics General HSC Q25] (4 marks) Paul invested money in a bank for 4 years. The stated interest rate on the account was 6.1% per annum compounded annually. This is equivalent to an effective simple interest rate of 6.68% per annum.

The formula Paul used to calculate the effective simple interest rate was:

$$E = \frac{(1+r)^n - 1}{n}$$

where

- r is the stated interest rate per period (expressed as a decimal)
- E is the effective simple interest rate per period (expressed as a decimal)
- n is the number of periods

Martha invested money in a different bank for 4 years. The stated interest rate on her investment was 6% per annum compounded monthly.

Martha thinks that she has a better deal than Paul. Do you agree? Justify your answer by comparing their effective simple interest rates.

Answer: Martha: 6.76% p.a. effective, receives a better deal

Example 7

[2018 WACE Mathematics Applications Q16] Natalia inherits a sum of money from her grandfather. She wishes to place it in a high-interest savings account.

She is considering the following two options:

- Account A: interest rate 4.40% per annum, compounded monthly
- Account B: interest rate 4.30% per annum, compounded daily.

(a) The effective annual interest rate for Account A is 4.49% (correct to two decimal places). Determine the effective annual interest rate for Account B. **1**

Natalia's bank offers her another account, C, with an interest rate of 4.50% per annum.

(b) Under what circumstances will this interest rate and the effective annual interest rate be the same? **1**

(c) Which account (A, B or C) should Natalia choose to maximise her savings? Explain your reasoning. **2**

Answer: (a) 4.39% (b) If interest is compounded annually (c) C

Further exercises

(A) **Ex 8C** (Pender et al., 2019a)

- Q2-17

(X1) **Ex 14C** (Pender et al., 2019b)

- Q8-19

Section 3

Annuities

Learning Goal(s)

Knowledge	**Skills**	**Understanding**
How to solve	Careful algebraic manipulation	Home loan, regular savings, car loan, superannuation

By the end of this section am I able to:

34.4 Solve problems involving financial decisions, including but not limited to a home loan, a savings account, a car loan or superannuation

Definition 7

Annuity a form of investment involving a series of *equal*, *periodic* contributions for a specified term; interest compounding at the end of each specified period.

- Forms of annuities:
  - Superannuation
  - Regular savings plans
  - Growth questions involving discrete units of time

3.1 Calculations by derivation using geometric series

Fill in the spaces

- Derivation of expressions will require the .... formula, applied ....

$$A = P(1+r)^n$$

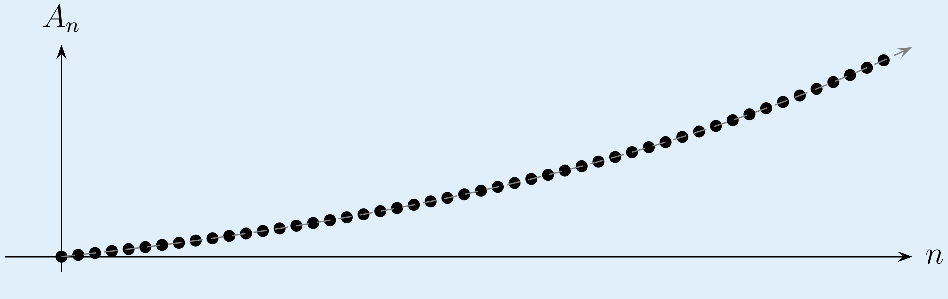
where

  - A is the final amount available.
  - P is the principal amount invested.
  - r is the normalised interest rate (to days/weeks/fortnights/months etc)
  - n is the number of compounding periods

Carefully read when the first deposit is made.

  - In calculations involving geometric series, usually the deposits are made a the .... of the year.

- General graph: increasing ....



  - More frequent .... will result in a faster .... of interest.

Excel functions to be aware of:

  - `fv`
  - `pmt`
  - `nper`
  - `pv`
  - `pduration`
  - `rate`
  - `fvschedule` - changing interest rates!

Example 8

A man invests $750 at the beginning of each year in a superannuation scheme. If the interest is paid at the rate of 8% p.a. on the investment, compounded annually, how much will his investment be worth after 20 years? **Answer:** $37 067.19

Steps

1. Write A_1, the amount in the account at the end of the first year.

$A_1 =$ ....

2. Write A_2 in terms of the new investment plus A_1, then simplify:

$A_2 =$ ....

$=$ ....

$=$ ....

3. Repeat, do so for A_3.

$A_3 =$ ....

$=$ ....

$=$ ....

4. Generalise to A_n:

$A_n =$ ....

5. Find sum of geometric progression, and apply sum of GP formula:

6. Solve to find A_{20}:

3.1.1 Alternate derivation

- Treat as a sum of n compound interest questions.

Important note

A sum of a geometric progression *always* appears.

Example 9

[**2007 2U HSC Q9**] Mr and Mrs Caine each decide to invest some money each year to help pay for their son's university education. The parents choose different investment strategies.

(i) Mr Caine makes 18 yearly contributions of \$1000 into an investment fund. He makes his first contribution on the day his son is born, and his final contribution on his son's seventeenth birthday. His investment earns 6% compound interest per annum. **3**

Find the total value of Mr Caine's investment on his son's eighteenth birthday.

(ii) Mrs Caine makes her contributions into another fund. She contributes \$1000 on the day of her son's birth, and increases her annual contribution by 6% each year. Her investment also earns 6% compound interest per annum. **2**

Find the total value of Mrs Caine's investment on her son's third birthday (just before she makes her fourth contribution).

(iii) Mrs Caine also makes her final contribution on her son's seventeenth birthday. Find the total value of Mrs Caine's investment on her son's eighteenth birthday. **1**

Example 10

[**2014 JRAHS 2U Trial**] Mitchell invests \$50 on the 1st of every month starting from January 2003 into a Superannuation Fund. Interest is paid at the rate of 5% p.a. compounded monthly. **Answer:** i. Show. ii. \$640 133.54 iii. \$794 748.98 iv. \$259.41

i. Show that his investment is worth \$85 923.96 by the end of 2044 **3**

ii. Mitchell's employer pays 9% of his monthly income into the same fund for the same amount of time. If Mitchell earns \$43 000 p.a. how much is his fund worth in total at the end of 2044? **2**

iii. Assuming that Mitchell and his boss pay as stated in parts i. and (ii), calculate the value of Mitchell's fund at the end of 2044 if the interest rate increases to 6% at the beginning of 2020. **3**

iv. Using the information from part (ii) and using the original interest rate only, find the amount that Mitchell must invest each month if he wishes to have a superannuation fund value of \$1 000 000 in total. **2**

Example 11

[**2014 CSSA 2U Trial**] Monica invests \$250 into an account at the Bank of Newton. She invests the money at the beginning of each month for n years. Interest is to be paid at a rate of 6% p.a. compounded monthly.

i. Show that the total value of her investment A_n at the end of n years is given by **2**

$$A_n = \$250\left(1.005 + 1.005^2 + \cdots + 1.005^{12n}\right)$$

ii. Find the value of the investment at the end of 7 years. **2**

iii. What single investment at the beginning of the 7 years would yield the same final value for Monica? You may assume interest is compounded monthly. **2**

Example 12

[2009 General Mathematics HSC Q17] Sally decides to put \$100 per week into her superannuation fund. The interest rate quoted is 8% per annum, compounded weekly.

Which expression will calculate the future value of her superannuation at the end of 35 years?

(A) $100\left\{\dfrac{\left(1+\frac{0.08}{52}\right)^{35}-1}{\frac{0.08}{52}}\right\}$ (C) $100\left\{\dfrac{\left(1+\frac{0.08}{52}\right)^{1820}-1}{\frac{0.08}{52}}\right\}$

(B) $100\left\{\dfrac{(1+0.08)^{35}-1}{0.08}\right\}$ (D) $100\left\{\dfrac{(1+0.08)^{1820}-1}{0.08}\right\}$

Important note

⚠ Students in General Mathematics were given a formula. Question can still be attempted by Mathematics Advanced students by quickly deriving the terms required.

Further exercises

Ⓐ **Ex 8D** (Pender et al., 2019a)

- Q1-21

(x1) **Ex 14D** (Pender et al., 2019b)

- Q3-11

3.1.2 Present value of an annuity

Example 13

[**2002 General Mathematics HSC Q15**] Calculate the present value of an annuity in which $1 200 is invested at the end of every year for ten years and interest is paid annually at a rate of 5% per annum. (Answer to the nearest dollar.)

(A) $1 922 (B) $9 266 (C) $15 093 (D) $30 654

Answer: (B)

Important note

- Students in General Mathematics were given a formula. Question can still be attempted by Mathematics Advanced students by quickly deriving the terms required. Refer to Definition 2 on page 5 for this question.
- $1 200 is *not* the present value of the annuity!

3.2 (S2) Calculations by table of interest factors

3.2.1 Future value interest factors

⚠ Read the tables very carefully!

⚠ By convention, the first deposit is made at the *end* of the first year!

Example 14

[**2005 Mathematics General HSC Q26**] Rod is saving for a holiday. He deposits \$3 600 into an account at the end of every year for four years. The account pays 5% per annum interest, compounding annually.

The table shows future values of an annuity of \$1.

Future values of an annuity of \$1

End of year	*Interest rate*				
	1%	**2%**	**3%**	**4%**	**5%**
1	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0200	2.0300	2.0400	2.0500
3	3.0301	3.0604	3.0909	3.1216	3.1525
4	4.0604	4.1216	4.1836	4.2465	4.3101
5	5.1010	5.2040	5.3091	5.4163	5.5256
6	6.1520	6.3081	6.4684	6.6330	6.8019
7	7.2135	7.4343	7.6625	7.8983	8.1420
8	8.2857	8.5830	8.8923	9.2142	9.5491

i. Use the table to find the value of Rod's investment at the end of four years. **2**

ii. How much interest does Rod earn on his investment over the four years? **2**

Answer: i. \$15 516.36 ii. \$1 116.36

- Identify the interest factor by a geometric series at the end of the 4th year.

Example 15

[**2009 General Mathematics HSC Q27**] The table shows the future value of a $1 annuity at different interest rates over different numbers of time periods.

Future values of a $1 annuity

Time Period	*Interest rate*				
	1%	*2%*	*3%*	*4%*	*5%*
1	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0200	2.0300	2.0400	2.0500
3	3.0301	3.0604	3.0909	3.1216	3.1525
4	4.0604	4.1216	4.1836	4.2465	4.3101
5	5.1010	5.2040	5.3091	5.4163	5.5256
6	6.1520	6.3081	6.4684	6.6330	6.8019
7	7.2135	7.4343	7.6625	7.8983	8.1420
8	8.2857	8.5830	8.8923	9.2142	9.5491

i. What would be the future value of a $5 000 per year annuity at 3% per annum for 6 years, with interest compounding yearly? **1**

ii. What is the value of an annuity that would provide a future value of $407 100 after 7 years at 5% per annum compound interest? **1**

iii. An annuity of $1 000 per quarter is invested at 4% per annum, compounded quarterly for 2 years. What will be the amount of interest earned? **3**

Example 16

[2020 Adv HSC Sample Q34/2019 Mathematics Standard 2 HSC Q42] (3 marks) The table shows the future values of an annuity of $1 for different interest rates for 4, 5 and 6 years. The contributions are made at the end of each year.

Future value of an annuity of $1

Years	*Interest rate per annum*			
	1%	*2%*	*3%*	*4%*
4	4.060	4.122	4.184	4.246
5	5.101	5.204	5.309	5.416
6	6.152	6.308	6.468	6.633

An annuity account is opened and contributions of $2 000 are made at the end of each year for 7 years.

For the first 6 years, the interest rate is 4% per annum, compounding annually.
For the 7th year, the interest rate increases to 5% per annum, compounding annually.

Calculate the amount in the account immediately after the 7th contribution is made.

Answer: 15 929.3

Example 17

[2021 Adv HSC Q25] (3 marks) A table of future value interest factors for an annuity of $1 is shown.

Table of future value interest factors

Number of periods	*Interest rate per period*				
	0.25%	0.5%	0.75%	1%	1.25%
2	2.0025	2.0050	2.0075	2.0100	2.0125
4	4.0150	4.0301	4.0452	4.0604	4.0756
6	6.0376	6.0755	6.1136	6.1520	6.1907
8	8.0704	8.1414	8.2132	8.2857	8.3589
10	10.1133	10.2280	10.3443	10.4622	10.5817

Simone deposits $1000 into a savings account at the end of each year for 8 years. The interest rate for these 8 years is 0.75% per annum, compounded annually.

After the 8th deposit, Simone stops making deposits but leaves the money in the savings account. The money in her savings account then earns interest at 1.25% per annum, compounded annually, for a further two years.

Find the amount of money in Simone's savings account at the end of ten years.

Answer: $8419.81

3.2.2 Other factor tables

Example 18

[**2016 Mathematics General 2 HSC Q28**] (2 marks) The table gives the contribution per period for an annuity with a future value of $1 at different interest rates and different periods of time.

Contribution per period for an annuity with a future value of $1

Number of periods	*Interest rate* (% per period)					
	0.25%	0.5%	0.75%	1%	1.25%	1.5%
6	0.1656	0.1646	0.1636	0.1625	0.1615	0.1605
12	0.0822	0.0811	0.0800	0.0788	0.0778	0.0767
18	0.0544	0.0532	0.0521	0.0510	0.0499	0.0488
24	0.0405	0.0393	0.0382	0.0371	0.0360	0.0349
30	0.0321	0.0310	0.0298	0.0287	0.0277	0.0266
36	0.0266	0.0254	0.0243	0.0232	0.0222	0.0212

Margaret needs to save $75 000 over 6 years for a deposit on a new apartment. She makes regular quarterly contributions into an investment account which pays interest at 3% pa.

How much will Margaret need to contribute each quarter to reach her savings goal?

Answer: $2 865

Further exercises

(A) **Ex 8F** - Click here for PDF

- All questions

(X1) **Ex 14F** - Click here for PDF

- All questions

Note these exercises are separate PDFs to the printed textbook.

Section 4

Loan repayments

4.1 Periodic deductions

Learning Goal(s)

Knowledge
Geometric sequences

Skills
Careful algebraic manipulations

Understanding
Home loans, regular savings, car loan, superannuation

By the end of this section am I able to:

34.3 Use geometric sequences to model and analyse practical problems involving exponential growth and decay

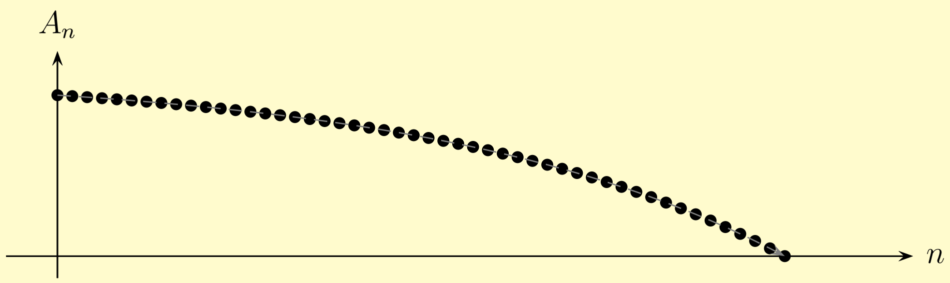
34.4 Solve problems involving financial decisions, including but not limited to a home loan, a savings account, a car loan or superannuation

4.1.1 Types of periodic deductions

- "Time"/Loan repayment involving reducible interest (compound interest analogue)
- Simultaneous growth/decay involving discrete units of time.

Laws/Results

- General graph: decreasing ....



  - More frequent .... will result in a faster .... of the entire loan.

Important note

- $A_n =$ .... when loan is repaid.
- Be aware of special conditions that apply to loan, such as multiple repayments prior to interest calculation.

4.2 Calculations by derivation using geometric series

4.2.1 Derivation of formula

Example 19

Natasha and Richard take out a loan of \$200 000 on 1st January 2002 to buy a house. Interest is charged at 12% p.a., compounded monthly, and they will repay the loan in monthly instalments of \$2 200.

(a) Find the amount owing at the end of n months.

(b) Find how long it takes to repay:

i the full loan, ii half the loan.

(c) How long would repayment take if they were able to pay \$2 500 per month?

(d) Why would instalments of \$1 900 per month never repay the loan?

Answer: (a) $A_n = 220\,000 - 1.01^n \times 20\,000$ (b) i. 20 yrs 1 mth ii. 15 yrs (c) 13 yrs 6 mths

Steps

(a) Find the amount owing at the end of n months

1. Write A_1, the amount in the account at the end of the first month.

$A_1 =$ ....

2. Write A_2 after having accumulated interest, based on A_1, but subtracting repayment:

$A_2 =$ ....

$=$ ....

$=$ ....

3. Repeat, do so for A_3.

$A_3 =$ ....

$=$ ....

$=$ ....

Steps

(a) Find the amount owing at the end of n months
(continued from previous page)

4. Generalise to A_n:

$A_n =$ ....

....

5. Find sum of geometric progression, and apply sum of GP formula:

$S_n =$ .... $=$ ....

$=$ ....

6. Solve to find A_n:

$A_n =$ ....

$=$ ....

$=$ ....

Important note

A sum of a geometric progression *always* appears.

Example 20

[2012 2U HSC Q15] Ari takes out a loan of \$360 000. The loan is to be repaid in equal monthly repayments, \$$M$, at the end of each month, over 25 years (300 months). Reducible interest is charged at 6% per annum, calculated monthly.

Let \$$A_n$ be the amount owing after the n-th repayment.

i. Write down an expression for the amount owing after two months, \$$A_2$. **1**

ii. Show that the monthly repayment is approximately \$2 319.50. **2**

iii. After how many months will the amount owing, A_n, become less than \$180 000? **3**

Example 21

[2000 2U Trial Q10] A store offers a loan of $5 000 on a computer for which it charges interest at the rate of 1% per month. As a special deal, the store does not charge interest for the first three months however, the first repayment is due at the end of the first month.

A customer takes out the loan and agrees to repay the loan over three years by making 36 equal monthly repayments of $\$M$. Let $\$A_n$ be the amount owing at the end of the n-th repayment.

i. Find an expression for A_3. **1**

ii. Show that $A_5 = (5\,000 - 3M)\,1.01^2 - M(1 + 1.01)$ **1**

iii. Find an expression for A_{36}. **2**

iv. Find the value of M. **2**

Example 22

[2003 2U HSC Q10] Barbara borrows \$120 000 to be repaid over a period of 25 years at 6% per annum reducible interest. Each year there are k regular repayments of \$$F$. Interest is calculated and charged just before each repayment.

i. Write down an expression for the amount owing after two repayments. **1**

ii. Show that the amount owing after n repayments is **2**

$$A_n = 120\,000\alpha^n - \frac{kF\left(\alpha^n - 1\right)}{0.06}$$

where $\alpha = 1 + \dfrac{0.06}{k}$.

iii. Calculate the amount of each repayment if the repayments are made quarterly (ie. $k = 4$). **2**

iv. How much would Barbara have saved over the term of the loan if she had chosen to make monthly rather than quarterly repayments? **2**

Example 23

[**2006 2U HSC Q8**] Joe borrows \$200 000 which is to be repaid in equal monthly instalments. The interest rate is 7.2% per annum reducible, calculated monthly.

It can be shown that the amount, $\$A_n$, owing after the n-th repayment is given by the formula:

$$A_n = 200\,000r^n - M\left(1 + r + r^2 + \cdots + r^{n-1}\right)$$

where $r = 1.006$ and $\$M$ is the monthly repayment (Do NOT show this).

i. The minimum monthly repayment is the amount required to repay the loan in 300 instalments. **3**

Find the minimum monthly repayment.

ii. Joe decides to make repayments of \$2 800 each month from the start of the loan. **2**

How many months will it take for Joe to repay the loan?

Example 24

[2021 Adv HSC Q29]

(a) On the day that Megan was born, her grandfather deposited \$5 000 into an account earning 3% per annum compounded annually. On each birthday after this, her grandfather deposited \$1 000 into the same account, making his final deposit on Megan's 17th birthday. That is, a total of 18 deposits were made. **2**

Let A_n be the amount in the account on Megan's n-th birthday, after the deposit is made.

Show that $A_3 = \$8\,554.54$

(b) On her 17th birthday, just after the final deposit is made, Megan has \$30 025.83 in her account. You are NOT required to show this. **3**

Megan then decides to leave all the money in the same account continuing to earn interest at 3% per annum compounded annually. On her 18th birthday, and on each birthday after this, Megan withdraws \$2 000 from the account.

How many withdrawals of \$2 000 will Megan be able to make?

Answer: $n = 20.249\cdots$ (20 withdrawals)

4.2.2 Additional questions

1. [**2009 2U HSC Q8**] One year ago Daniel borrowed \$350 000 to buy a house. The interest rate was 9% per annum, compounded monthly. He agreed to repay the loan in 25 years with equal monthly repayments of \$2 937.

i. Calculate how much Daniel owed after his first monthly repayment. **1**

ii. Daniel has just made his 12th monthly repayment. He now owes \$346 095. The interest rate now decreases to 6% per annum, compounded monthly. **3**

The amount, $\$A_n$, owing on the loan after the n-th monthly repayment is now calculated using the formula

$$A_n = 346\,095 \times 1.005^n - 1.005^{n-1}M - \cdots - 1.005M - M$$

where $\$M$ is the monthly repayment, and $n = 1, 2, \ldots, 288$. (Do NOT prove this formula).

Calculate the monthly repayment if the loan is to be repaid over the remaining 24 years (288 months).

iii. Daniel chooses to keep his monthly repayments at \$2 937. Use the formula in part (ii) to calculate how long it will take him to repay the \$346 095. **3**

iv. How much will Daniel save over the term of the loan by keeping his monthly repayments at \$2 937, rather than reducing his repayments to the amount calculated in part (ii)? **1**

Answers

1. (i) \$349 688 (ii) \$2 270.31 (iii) $n = 178.3733\cdots \approx$ 14 yrs 11 mths (iv) \$129 966.90 or \$128 126.90 (if using 179 mths)

4.2.3 Non-financial situations

Example 25

[**2020 Mathematics Advanced Sample HSC Q36**] An island initially has 16 100 trees. The number of trees increases by 1% per annum. The people on the island cut down 1 161 trees at the end of each year.

(a) Show that after the first year there are 15 100 trees remaining. **1**

(b) Show that at the end of 2 years the number of trees remaining is given by the expression **2**

$$T_2 = 16\,100 \times (1.01)^2 - 1\,161(1 + 1.01)$$

(c) Show that at the end of n years the number of trees remaining is given by the expression **2**

$$T_n = 116\,100 - 100\,000 \times (1.01)^n$$

(d) For how many years will the people on the island be able to cut down 1 161 trees annually? **1**

Answer: $\approx$ 15 years

Example 26

[1998 2U HSC Q10] A fish farmer began business on 1 January 1998 with a stock of 100 000 fish. He had a contract to supply 15 400 fish at a price of $10 per fish to a retailer in December each year. In the period between January and the harvest in December each year, the number of fish increases by 10%.

Answer: i. 88 660 ii. Show iii. $169 400 iv. 7 yrs, 12/2005

i. Find the number of fish just after the second harvest in December 1999. **1**

ii. Show that F_n, the number of fish just after the n-th harvest, is given by **2**

$$F_n = 154\,000 - 54\,000(1.1)^n$$

iii. When will the farmer have sold all his fish, and what will his total income be? **2**

iv. Each December the retailer offers to buy the farmer's business by paying $15 per fish for his entire stock. When should the farmer sell to maximise his total income? **2**

Further exercises

Ⓐ **Ex 6E** (Pender, Sadler, Shea, & Ward, 2009)

- Q5-13

ⓧ1 **Ex 14E** (Pender et al., 2019b)

- Q5-13

4.3 S2 Calculations by table of interest factors

Definition 8

The **present value of a loan** is the original loan balance.

Definition 9

The **present value interest factor** ($PVIF$) **for a loan** is the .... of the regular loan repayment amount to pay off the loan plus reducible interest.

$$A_0 = M \times PVIF$$

where

- A_0 is the original loan balance.
- $PVIF$ is the *present value interest factor*.
- M is the regular repayment amount.

Example 27

[2015 Independent General 2 Trial HSC Q30] (with modifications) Jackson has a personal loan of \$15 000 and has to repay this loan in equal monthly payments over 4 years. The interest rate on Jackson's loan is 7.8% pa.

Additional insertion

(a) If M is the monthly repayment amount, show that **3**

$$A_n = 15\,000 \times \left(\frac{2\,013}{2\,000}\right)^n - \frac{2000M}{13}\left(\left(\frac{2\,013}{2\,000}\right)^n - 1\right)$$

(b) When the loan is repaid, $A_{48} = 0$. **2**

By changing the subject of the A_{48} expression to 15 000, show that the present value interest factor is approximately 41.1199.

Original question The following table shows the present value interest factors for reducing balance loans at various monthly interest rates (r) over different time periods (N).

	r					
N	0.0060	0.0065	0.0070	0.0075	0.0080	0.0085
45	39.3341	38.9074	38.4871	38.0732	37.6655	37.2638
46	40.0935	39.6497	39.2126	38.7823	38.3586	37.9413
47	40.8484	40.3871	39.9331	39.4862	39.0462	38.6131
48	41.5988	41.1199	40.6486	40.1848	39.7284	39.2792
49	42.3448	41.8478	41.3590	40.8782	40.4051	39.9398
50	43.0862	42.5711	42.0646	41.5664	41.0765	40.5947

i. Write down the present value interest factor from the table associated with Jackson's loan. **1**

ii. Calculate the interest that Jackson will pay over the term of his loan. **2**

Example 28

[**2016 NBHS General 2 Trial HSC Q26**] The table shows *present value interest factors* for some monthly interest rates (r) and loan terms in months (N).

Each number in the table below is the **present value** of an annuity of $1 at the end of each period.

Term	**Monthly interest rate (as a decimal)**						
(mths)	0.004	0.0045	0.005	0.0055	0.006	0.0065	0.007
106	86.26	84.15	82.12	80.16	78.26	76.43	74.66
107	86.91	84.77	82.71	80.72	78.79	76.93	75.13
108	87.56	85.39	83.29	81.27	79.32	77.43	75.60
109	88.20	86.00	83.87	81.82	79.84	77.92	76.07
110	88.85	86.61	84.45	82.37	80.35	78.41	76.53
111	89.49	87.22	85.03	82.91	80.87	78.90	77.00
112	90.13	87.82	85.60	83.45	81.38	79.38	77.45
113	90.77	88.43	86.17	83.99	81.89	79.86	77.91
114	91.40	89.03	86.73	84.53	82.40	80.34	78.36
115	92.03	89.62	87.30	85.06	82.90	80.82	78.81
116	92.66	90.22	87.86	85.59	83.40	81.29	79.25
117	93.29	90.81	88.42	86.11	83.89	81.76	79.70
118	93.91	91.40	88.97	86.64	84.39	82.22	80.13
119	94.54	91.98	89.52	87.16	84.88	82.68	80.57
120	95.16	92.57	90.07	87.68	85.37	83.14	81.00
121	95.77	93.15	90.62	88.19	85.85	83.60	81.43
122	96.39	93.72	91.16	88.70	86.33	84.05	81.86
123	97.00	94.30	91.71	89.21	86.81	84.50	82.28

Mr Anderson borrows $65 000 for home improvements. He repays the loan with monthly repayments over 10 years. He is charged 6% p.a. interest.

i. Calculate the amount of his monthly instalment, correct to the nearest cent. **1**

ii. How much less interest would he pay if he took the loan over 9 years instead of 10? **2**

Answer: i. 721.66 ii. 2 314.92

Example 29

[2015 Mathematics General 2 HSC Q30] The table gives the present value interest factors for an annuity of \$1 per period, for various interest rates (r) and numbers of periods (N).

Table of present value interest factors

N \ r	Interest rate per period (as a decimal)				
	0.0075	0.0080	0.0085	0.0090	0.0095
70	54.30462	53.43960	52.59397	51.76724	50.95891
71	54.89293	54.00754	53.14226	52.29657	51.46995
72	55.47685	54.57097	53.68593	52.82118	51.97618
73	56.05643	55.12993	54.22502	53.34111	52.47764
74	56.63169	55.68446	54.75957	53.85641	52.97438

i. Oscar plans to invest \$200 each month for 74 months. His investment will earn interest at the rate of 0.0080 (as a decimal) per month. **1**

Use the information in the table to calculate the present value of this annuity.

ii. Lucy is using the same table to calculate the loan repayment for her car loan. Her loan is \$21 500 and will be repaid in equal monthly repayments over 6 years. The interest rate on her loan is 10.8% per annum. **2**

Calculate the amount of each monthly repayment, correct to the nearest dollar.

Answer: i. \$11136.89 ii. \$407

Important note

⚠ The present value of an *annuity* (See Example 13 on page 24) is subtly different to that of a *loan* (Definition 8 on page 45).

Further exercises

(A) **Ex 8G** - Click here for PDF

- All questions

(x1) **Ex 14G** - Click here for PDF

- All questions

Note these exercises are separate PDFs to the printed textbook.

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, \; r \neq 1$$

$$S = \frac{a}{1-r}, \; |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

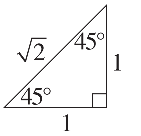
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$

$A = \frac{1}{2}ab\sin C$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

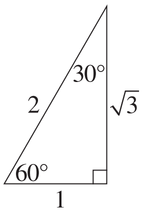


$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \frac{1}{2}r^2\theta$



Trigonometric identities

$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$\cos A = \frac{1 - t^2}{1 + t^2}$

$\tan A = \frac{2t}{1 - t^2}$

$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$

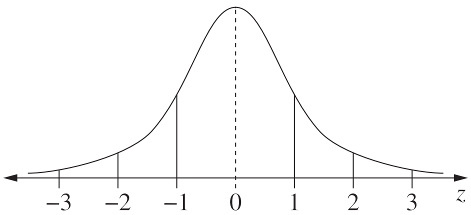
$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \frac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \le x) = \int_a^x f(x)\,dx$

$P(a < X < b) = \int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n - r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \quad P(X = x)$

$= \binom{n}{x} p^x (1 - p)^{n - x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2 + y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

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